Dynamic Consistency and Ambiguous Communication*

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Abstract

In most models of ambiguous communication, a Sender can only benefit from ambiguous communication if the Receiver behaves dynamically inconsistently. A dynamically inconsistent Receiver might not follow his ex-ante optimal plan after observing an ambiguous message. This paper proposes a novel approach to analyze ambiguous communication by studying dynamically consistent behavior in games with ambiguous strategies. We show that gains from ambiguous communication can be maintained even if players behave dynamically consistently. To achieve this, we define rectangularity, a condition on beliefs that ensures dynamically consistent behavior, for settings where ambiguity arises due to ambiguous strategies. Then, we analyze a Perfect Bayesian Equilibrium in an ambiguous persuasion setting. In this equilibrium, ambiguous communication outperforms standard Bayesian communication even if the Receiver behaves dynamically consistently. Finally, we extend our analysis to settings with ambiguous communication in cheap talk and mechanism design.

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1 Introduction

Ambiguous communication is widely used, e.g., in political and business communication as well as in legalistic or technical writings. One example is fedspeak, a term introduced to describe the wordy and vague language used by chairs of the Federal Reserve Board. Furthermore, firms, lobbyists, and politicians spend an extensive amount of time and resources to elaborate on communication strategies – see, for example, McCloskey and Klamer (1995). Thus, even if ambiguous communication increases uncertainty, it seems to play an essential role in strategic communication.

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The present paper studies the role of ambiguous communication and its strategic use for dynamically consistent and ambiguity averse players. It has been shown that one or even all players can benefit from ambiguous communication in mechanism design (Bose and Renou (2014)), cheap talk (Kellner and Le Quement (2018)), and persuasion (Beauchêne et al. (2019)). However, in this literature, ambiguous communication is profitable only if players behave dynamically inconsistently. Thus, the effect of ambiguity aversion and dynamically inconsistent behavior on the gain of ambiguous communication cannot be separated in these settings.

This paper proposes a novel approach to studying dynamically consistent behavior in games with ambiguous communication. In our setting, ex-ante, the Receiver considers his knowledge about the information structure and how ambiguous information may manipulate him. Then, we show that the gain of ambiguous communication does not rely on dynamically inconsistent behavior. To study dynamically consistent behavior, we define rectangularity – a condition on the belief set – that implies dynamically consistent behavior. Further, we propose and examine a Perfect Bayesian Equilibrium with rectangular beliefs in settings with strategic ambiguous communication.

In contrast to risk, ambiguity captures uncertainty that cannot be modeled by a single probability measure. One way to model ambiguity averse preferences is the maxmin expected utility model (MEU), proposed by Gilboa and Schmeidler (1989). An agent with MEU preferences faces a set of possible beliefs instead of one single belief and maximizes his worst-case expected utility. Further, the literature proposes different updating rules for ambiguity averse agents. We will follow the prior-by-prior Bayesian updating approach (or full Bayesian updating), which assumes that the set of updated beliefs consists of all Bayesian updates of the ex-ante belief set.

Almost all ambiguity averse preferences and updating rules proposed by the literature may lead to dynamically inconsistent behavior. An agent behaves dynamically inconsistently if he does not follow his ex-ante optimal plan after receiving information and updating his beliefs. Roughly speaking, new information can lead to a change in the worst-case belief and, therefore, changes the optimal strategy. Dynamically inconsistent behavior makes it impossible to use standard equilibrium concepts such as Perfect Bayesian Nash Equilibrium or Sequential Equilibrium. Furthermore, it leads to problems, e.g., in analyzing welfare and value of information, since ex-ante and ex-post decisions are not comparable.

Most of this paper focuses on ambiguous persuasion. However, we show that the same methods and technics can be applied analogously for ambiguous communication in mechanism design and cheap talk. Similar to Beauchêne et al. (2019), we introduce ambiguity in the standard Bayesian persuasion setting of Kamenica and Gentzkow (2011) by allowing the Sender to choose a set of communication devices. Each communication device can generate a message that reveals information about an unknown (risky) state $\omega \in \Omega$. Sender and Receiver only observe one message without knowing which communication device generated the message. Thus, an ambiguous communication device implies an ambiguous interpretation of the observed message and, therefore, ambiguity about the state ω.

To ensure dynamically consistent behavior, we define rectangularity for settings with ambiguous communication. In these settings, ambiguity arises endogenously due to the ambiguous strategy of the Sender. Intuitively, rectangularity allows players to take into account their knowledge about the information structure and potential future worst-case beliefs. Given the Sender's strategy, a rational Receiver knows which ambiguous message he can observe and how this influences his interim worst-case belief. Formally, rectangularity implies a subjective ex-ante belief set that considers discrepancies between the future and current worst-case beliefs.

To formalize rectangular beliefs, we define beliefs on a state space that depends on the (risky) state ω and the messages. This state space takes the dependence of the ambiguous signal and the ex-ante risky state into account and allows for a non-singleton ex-ante belief set. First, we show that one can restrict the message set to straightforward messages and synonyms. A message set consists of straightforward messages if it only contains recommendations on which action the Receiver should choose. A synonym m' of a message m is a message that induces the same posterior belief or best response of the Receiver as the message m. This result generalizes the well-known Proposition 1 of Kamenica and Gentzkow (2011), which states that one can restrict the message set, without loss of generality, to straightforward messages in the ambiguous setting.

Then, we define rectangular beliefs over the general state space of straightforward messages and states. Given rectangular beliefs, we can extend the usual definition of a Perfect Bayesian Equilibrium to settings with ambiguous communication. We examine a Perfect Bayesian Equilibrium. Further, we show that all results of Beauchêne et al. (2019) can be extended to a Perfect Bayesian Equilibrium with rectangular beliefs. Hence, the gain of ambiguous communication does not rely on dynamically inconsistent behavior. Ambiguous communication is profitable due to ambiguity averse preferences and not due to dynamically inconsistent behavior.

The paper is organized as follows: First, we review the related literature. In Section 2, we formulate the ambiguous persuasion model and give an example that illustrates the gain of an ambiguous strategy and the dynamically inconsistent behavior. Further, Section 2.3 generalizes Proposition 1 of Kamenica and Gentzkow (2011) and defines rectangular beliefs. In Section 3, we define Perfect Bayesian Equilibria under rectangular beliefs and compare our results to Beauchêne et al. (2019). In Section 4, we discuss rectangular beliefs in an ambiguous cheap talk setting and a mechanism design setting with ambiguous communication. Finally, Section 5 concludes and discusses extensions and related issues.

1.1 Related Literature

This paper contributes to the literature on ambiguous communication. Among others, Beauchêne et al. (2019), Kellner and Le Quement (2018), and Bose and Renou (2014) study ambiguous communication in persuasion, cheap talk, and mechanism design. In all three settings, ambiguity arises endogenously due to the ambiguous communication of the

Sender.[1] However, in all these papers, ambiguous communication leads to new equilibria only if players behave dynamically inconsistently. For example, Beauchêne et al. (2019) claim that there is no gain of ambiguous persuasion compared to Bayesian persuasion if the players behave dynamically consistently.[2]

The present paper proposes a novel approach to studying ambiguous communication. Our setting differs from the previous paper by allowing players to consider the discrepancy between their current and future worst-case beliefs.

To implement dynamically consistent behavior, we extend the concept of rectangular beliefs to settings where ambiguity arises due to ambiguous strategies. Epstein and Schneider (2003), Sarin and Wakker (1998), and Riedel et al. (2018) define rectangularity for decision theoretical settings with a fixed information structure. Pahlke (2022) generalizes the concept of rectangularity to multi-stage games with ambiguity about states but non-ambiguous strategies. However, ambiguous beliefs arise in these settings due to exogenous ambiguity about states or types. In the present paper, ambiguous beliefs arise endogenously due to ambiguous strategies. To our knowledge, Muraviev et al. (2017) is the only work analyzing rectangularity for strategic use of ambiguity. However, they only study the relationship between mixed and behavior strategies and do not define an equilibrium concept. Thus, the present paper defines a dynamically consistent equilibrium concept for games with ambiguous strategies for the first time in this literature.

Concurrent with this work, Cheng (2021) analyzes dynamic consistency for ambiguous persuasion. However, instead of rectangularity, he uses the updating rules of Hanany and Klibanoff (2007). Roughly speaking, these updating rules imply dynamically consistent behavior by assuming that players only update beliefs consistent with the ex-ante worst-case belief. Therefore, the ex-ante optimal choice of a player becomes interim optimal. Cheng (2021) shows that players using the updating rules of Hanany and Klibanoff (2007) can not gain from ambiguous persuasion. We discuss the relation between our work and Cheng (2021) in more detail in Section 5.

Finally, this paper is related to the literature on the value of ambiguous information. Li (2020) and Hill (2020) theoretically study and define the value of ambiguous information. In contrast to Li (2020), we show that ambiguous communication can imply a negative or positive value of information. We discuss the relation to these approaches in more detail in Section 5. Further, the value of ambiguous communication has been studied experimentally. Kops and Pasichnichenko (2022) and Ortoleva and Shishkin (2021) find heterogeneous results. While Kops and Pasichnichenko (2022) report a negative value of information for ambiguity averse players, Ortoleva and Shishkin (2021) cannot find a correlation between the negative value of information and ambiguity aversion.

[1]Additionally to ambiguous communication, Bose and Renou (2014) allow for exogenous ambiguity about the state. We discuss a similar generalization for persuasion and cheap talk in Section 5.

[2]See Proposition 5 of Beauchêne et al. (2019).

2 Model

The basic setting follows the model of Beauchêne et al. (2019), henceforth BLL, which extends the standard Bayesian persuasion setting of Kamenica and Gentzkow (2011) by an ambiguous communication device.

2.1 Setting

As the standard Bayesian persuasion, an ambiguous persuasion game consists of a Sender (she) and a Receiver (he). The utility of both players depends on the state $\omega \in \Omega$ and action $a \in A$ chosen by the Receiver, where Ω and A are compact subsets of the Euclidean space. We denote with $u(a,\omega)$ and $\nu(a,\omega)$ the utility of Receiver and Sender, respectively. Further, Sender and Receiver have maxmin preferences á la Gilboa and Schmeidler (1989), i.e., they maximize their worst-case expected utility.

Ex-ante, the state ω is unknown, and both players have the same prior state belief $p_0 \in \Delta\Omega$, i.e., ex-ante exists no ambiguity about the state.[3] The Sender tries to persuade the Receiver by choosing a signal that reveals information about the state. A signal consists of a finite set of signal realizations or messages M and a set of communication devices $\Pi = \{\pi_k\}_{k\in K}$.[4] Each communication device is a distribution over the set of messages M for each $\omega \in \Omega$, i.e., $\pi_k(\cdot|\omega) \in \Delta M$ for all $\omega \in \Omega$. As BLL, we assume that the π_k's have common support for all $k \in K$.

Thus, the only difference to the standard Bayesian persuasion setting is that the Sender chooses a set of communication devices instead of one communication device. Which of the communication devices generates the observed message is ambiguous to both players. After observing a message m, the Receiver updates his prior state belief prior-by-prior using Bayes' rule. Since he does not know which communication device generated the message, he updates p_0 with respect to each communication device π_k. This procedure leads to the following set of posterior state beliefs after observing the message $m \in M$

$$P_m = \left\{ p_m^{\pi_k}(\cdot) \in \Delta\Omega : p_m^{\pi_k}(\cdot) = \frac{p_0(\cdot)\pi_k(m|\cdot)}{\int_\Omega p_0(\omega)\pi_k(m|\omega)\,\mathrm{d}\omega}, \pi_k \in \Pi \right\}. \tag{1}$$

Then, after observing message m, the Receiver maximizes his interim worst-case expected utility

$$U(a, P_m) = \min_{p_m \in P_m} \mathbb{E}_{p_m}(u(a,\omega)). \tag{2}$$

As usual in the persuasion literature, we assume that the Receiver chooses the Sender's

[3]Our definition of belief differs from the one of BLL. To avoid confusion, we use the term state belief whenever we refer to beliefs in the sense of BLL.

[4]Please note that we deviate from the model of BLL by defining Π as the set of communication devices. BLL define Π as the convex hull of the set of communication devices. Since Sender and Receiver have maxmin preferences, the minimization problems over $\{\pi_k\}$ or $\mathrm{co}\big(\{\pi_k\}\big)$ coincide.

preferred action if he faces multiple maximizers. We denote with $\hat{a}_m$ the (Sender preferred) best response of the Receiver after observing the message m.

The Sender chooses the signal (M, Π) that maximizes her ex-ante worst-case expected utility

$$\sup_{(M,\Pi)} \min_{\pi \in \Pi} \mathbb{E}_{p_0}\Big[\mathbb{E}_\pi\left[\nu(\hat{a}_m, \omega)|\omega\right]\Big].$$

2.2 Dynamic Inconsistency

It is well-known that ambiguity might lead to dynamically inconsistent behavior as worst-case beliefs change over time. In our setting, the Sender only chooses an action at the ex-ante stage. Thus, she can never behave dynamically inconsistently. However, the interim best response of the Receiver is, in general, not ex-ante optimal. Intuitively, ex-ante the Receiver can hedge against ambiguity by playing a constant strategy. However, after observing the realized message, this strategy is no longer optimal.

The following example illustrates that ambiguity can lead to a higher expected payoff for the Sender. Furthermore, we show that the interim equilibrium strategy of the Receiver is not ex-ante ante optimal.

Example 1. *Consider a firm (Receiver) that wants to launch a product in a new country or geographical area. However, introducing the product to the new market is risky due to legal restrictions, e.g., data protection regulations. Therefore, the firm asks the legal department (Sender) to prepare a report about the risk of launching the product in the new area.*

Suppose there are two possible states: launching the product can be profitable ω_p or non-profitable ω_n. The firm can decide to launch l or suspend s the product. He only wants to launch the product if the state is profitable and suspend it otherwise. However, the legal department worries about additional work generated by launching the product in a new area. Therefore, she prefers that the product is always suspended. The payoffs of the Sender and Receiver are summarized in Figure 1.

	ω_n	ω_p
s	$(1,1)$	$(1,0)$
l	$(0,-1)$	$(0,1)$

Figure 1: Payoffs (S, R)

The firm and the legal department have a common ex-ante state belief $p_0 = \mathbb{P}(\omega_n) = 0.2$. Thus, without any additional information, the firm always launches the product. If the legal department fully reveals the state, the firm would suspend the product whenever it is non-profitable, which occurs with a probability of 0.2.

The optimal Bayesian persuasion generates either the message p or n with probabilities as described in Figure 2.

π	ω_n	ω_p
p	0	0.5
n	1	0.5

Figure 2: Optimal Bayesian Persuasion

Given this communication device, the Receiver suspends the product if he observes the message n and launches the product if he observes the message p. Thus, given the optimal Bayesian persuasion, the product is suspended with a probability of 0.6.

However, if the Sender uses an ambiguous communication device, she can still increase the probability of suspending. Suppose the Sender can create ambiguity by designing two communication devices π and π'. As before, the Receiver only observes one message without knowing which communication devices generated the message. Consider the two communication devices in Figure 3.

π	ω_n	ω_p
p	0	1
n	1	0

π'	ω_n	ω_p
p	1	0
n	0	1

Figure 3: Optimal Ambiguous Persuasion

Suppose the Receiver observes the message p. If this message was generated by π his updated belief is $\mathbb{P}^{\pi}(\omega_g|p) = 1$. But if the message was generated by π' his updated belief is $\mathbb{P}^{\pi'}(\omega_g|p) = 0$. Thus, he only knows that the updated belief is either 0 *or* 1. *Since he does not know which communication device generated the message, he faces ambiguity and maximizes his worst-case expected utility. Launching the product generates a worst-case utility of* -1, *whereas suspending leads to a worst-case utility of* 0. *Thus, the firm prefers to suspend the product. Similarly, if the firm observes the message n, his updated belief is again either* 0 *or* 1, *and he suspends the product. Thus, both messages imply worst-case beliefs such that the firm always suspends the product.*[5]

To sum up, without additional information, the expected utility of the Sender is 0.2, *the best Bayesian persuasion implies an expected utility of* 0.6, *and the best ambiguous*

[5]The communication devices used in this example create as much ambiguity as possible. However, as we show in Section B.1 in the Appendix, the same results can be obtained with less extreme communication devices.

persuasion leads to an expected utility of 1*. Thus, the Sender gains from ambiguous communication compared to Bayesian persuasion.*

However, the strategy 'always suspending' is not ex-ante optimal for the Receiver. His ex-ante worst-case expected utility of suspending is

$$\begin{aligned}&\min\left\{\mathbb{P}^{\pi}(n)\mathbb{P}^{\pi}(\omega_n|n)+\mathbb{P}^{\pi}(p)\mathbb{P}^{\pi}(\omega_n|p),\mathbb{P}^{\pi'}(n)\mathbb{P}^{\pi'}(\omega_n|n)+\mathbb{P}^{\pi'}(p)\mathbb{P}^{\pi'}(\omega_n|p)\right\}\\=&\min\left\{\mathbb{P}^{\pi}(n),\mathbb{P}^{\pi'}(p)\right\}=0.2\end{aligned}$$

and his ex-ante worst-case expected utility of launching the product is

$$\begin{aligned}&\min\left\{\mathbb{P}^{\pi}(n)\mathbb{P}^{\pi}(\omega_n|n)(-1)+\mathbb{P}^{\pi}(p)\mathbb{P}^{\pi}(\omega_p|p),\mathbb{P}^{\pi'}(n)\mathbb{P}^{\pi'}(\omega_p|n)+\mathbb{P}^{\pi'}(p)\mathbb{P}^{\pi'}(\omega_n|p)(-1)\right\}\\=&\min\left\{\mathbb{P}^{\pi}(n)(-1)+\mathbb{P}^{\pi}(p),\mathbb{P}^{\pi'}(p)(-1)+\mathbb{P}^{\pi'}(n)\right\}=0.6.\end{aligned}$$

Thus, even considering the worst-case analysis at the ex-ante stage, the firm would be better off if he always launches the product. But this requires a strong commitment device. After observing any message, the worst-case belief implies that launching the product is no longer optimal.

2.3 Dynamically Consistent Beliefs

In our setting, ambiguity arises due to the ambiguous communication device. Ambiguous interim beliefs only occur due to the combination of a risky state and an ambiguous signal. In the previous example, the interim best response of the Receiver is, in general, not ex-ante optimal. Intuitively, ex-ante the Receiver can hedge against ambiguity by ignoring the message and playing the same action for all messages. Ignoring the message and playing a constant action requires a strong commitment device for the Receiver. In many applications, such a strong commitment device is unavailable, and a Receiver might fail to commit to the ex-ante optimal action. However, if the Receiver is aware of the information structure and the lack of a commitment device, he might consider his inconsistent beliefs. Knowing that his worst-case belief at the interim stage will contradict his worst-case belief at the ex-ante stage, he might get skeptical about his ex-ante belief and the relation between ambiguous messages and the risky state.

Consider the following two situations at the ex-ante stage:

1) The Receiver does not observe any message. All information about the state $\omega \in \Omega$ is represented by p_0.

2) As in situation 1), the Receiver knows p_0. Additionally, he knows that he will receive an ambiguous message before deciding.

In the first situation, the Receiver knows there will be no additional information. Hence, he chooses his optimal action, given the expected utility with respect to p_0. In the second situation, ex-ante, the Receiver has the same information about the state as in situation 1). However, he knows he will receive additional but ambiguous information before deciding. Furthermore, he knows this ambiguous information influences his interim beliefs and, therefore, his best response. A rational player should consider this knowledge about a game's information structure when deciding at the ex-ante stage. Rectangularity takes the interplay of the prior state belief p_0 and the knowledge about the information structure into account and, therefore, ensures dynamically consistent behavior.

We show that defining beliefs over a more general state space allows the definition of non-singleton rectangular belief sets. Then, given a rectangular belief set, the Receiver behaves dynamically consistently, and the consistent planning equilibrium of BLL is an ex-ante and interim optimal and, therefore, a Perfect Bayesian Equilibrium.

2.3.1 Straightforward Messages

In any persuasion setting, the set of messages M is part of the Sender's strategy. In the Bayesian persuasion setting, Kamenica and Gentzkow (2011) call a signal straightforward if $M \subseteq A$. They show that one can restrict to straightforward signals in a Bayesian persuasion setting without loss of generality. More precisely, for any signal, a straightforward signal exists that leads to the same expected utility of the Sender in equilibrium. The next proposition generalizes this result to ambiguous persuasion. It shows that the Sender chooses without loss of generality $M = A \cup \tilde{A}$ where $\tilde{A}$ is a duplicated set of A such that there exists a bijection $b(\cdot)$ between A and $\tilde{A}$. Given this result, we can define ex-ante beliefs on the general state space $\Omega \times (A \cup \tilde{A})$.

Proposition 1. *Let* $(M, \Pi) \in \operatorname{argsup} \min_{\pi \in \Pi} \mathbb{E}_{p_0} \left[\mathbb{E}_{\pi} \left[\nu(\hat{a}_m, \omega) | \omega \right] \right]$. *Let* $\tilde{A}$ *be such that there exist a bijection* $b(\cdot) : A \to \tilde{A}$ *between* A *and* $\tilde{A}$. *Then, there exist* (M', Π') *with* $M' = A \cup \tilde{A}$ *and* $\Pi' = \{\pi'_1, \pi'_2\}$ *such that* (M', Π') *generates the same value for the Sender as* (M, Π).

The intuition of the proposition is as follows. BLL show that ambiguous persuasion increases the value for the Sender compared to Bayesian persuasion only if the Sender uses a signal with synonyms. Synonyms are messages that copy the meaning of another message, i.e., they induce the same posterior state belief set or best response of the Receiver. Intuitively, the Sender uses synonyms to hedge himself against ambiguity. Furthermore, they show that for any ambiguous signal, one can find an ambiguous signal which consists only of two communication devices and leads to the same value. Hence, to use straightforward messages as in Kamenica and Gentzkow (2011), we have to duplicate the message space to allow for synonyms, and duplication is enough to generate the same value as any ambiguous signal. Thus, without loss of generality, we can assume that $M = A \cup \tilde{A}$. The detailed proof can be found in Section A in the Appendix.

Due to the assumption that all π_k have common support on M, Sender's strategy (M, Π) is completely characterized by Π. For the rest of the paper, we will use the term strategy of the Sender for such a Π. Furthermore, we denote with $\text{supp}(\Pi) = \text{supp}(\pi_k(\cdot|\omega)) \subset A \cup \tilde{A}$ the support of $\pi_k \in \Pi$ for all $k \in K$.

2.3.2 Rectangular Beliefs

Given the results from the previous section, we can define beliefs over the general state space $\Omega \times (A \cup \tilde{A})$. Defining rectangularity on the set of joint beliefs about the payoff-relevant state and messages is essential to maintain ambiguity and dynamic consistency.[6] In our setting, ambiguity arises due to ambiguous messages, but uncertainty only exists because of the risky state. These two sources of uncertainty depend on and influence each other. Considering joint beliefs allows the Receiver to take into account both sources of uncertainty and their relation.

Definition 1. *For a strategy* Π *of the Sender, we define the* ***set of ex-ante beliefs*** *of the Receiver as*

$$\Phi^0_\Pi = \Bigg\{ \rho^k \in \Delta(\Omega \times (A \cup \tilde{A})) : \exists \pi_k \in \Pi \ s.t.$$
$$\rho^k(\omega, m) = \begin{cases} p_0(\omega)\pi_k(m|\omega) & \text{if } m \in supp(\Pi) \\ 0 & \text{otherwise} \end{cases} \Bigg\}.$$

Note that the strategy of the Sender generates the information structure of the persuasion games. Therefore, it has to influence the joint belief over states and messages, and Φ^0_Π depends on Π.

At the interim stage, the Receiver observes a message $m \in \text{supp}(\Pi)$. The information structure at the ex-ante stage ($t = 0$) and interim stage ($t = 1$) can be represented by the following partitions

$$\mathcal{F}_0 = \Omega \times (A \cup b(A)),$$
$$\mathcal{F}_1 = \Big\{ \{\Omega \times m\}_{m \in A \cup b(A)} \Big\}.$$

Then, given an observation $\hat{m} \in \text{supp}(\Pi)$ the Receiver updates his ex-ante belief set prior-by-prior using Bayes' formula, i.e., he updates each prior belief in Φ^0_Π with Bayes' formula. We denote the updated belief after observing message $\hat{m} \in \text{supp}(\Pi)$ by $\rho^k_{\hat{m}}$. Then,

$$\rho^k_{\hat{m}}(\omega, m) = \frac{p_0(\omega)\pi_k(m|\omega)}{\int_\Omega p_0(\omega')\pi_k(m|\omega')\, \mathrm{d}\omega'}$$

if $m = \hat{m}$ and zero otherwise. The set of updated beliefs given $\hat{m} \in \text{supp}(\Pi)$ is denoted by

$$\text{Bay}(\Phi^0_\Pi|\hat{m}) = \{\rho^k_{\hat{m}} \text{ with } \pi_k \in \Pi\}.$$

[6] BLL define a version of rectangularity for beliefs on the payoff-relevant state. They show that their definition of rectangularity implies dynamic consistency but non-ambiguous ex-ante beliefs.

Remark 1. *Note that* $\rho^k_m((\omega,m)) = p^{\pi_k}_m(\omega)$ *as defined in Equation* (1) *for all* ω. *Therefore, the Receivers' maximization problem at the interim stage, given our definition of beliefs, coincides with the maximization problem of BLL of Equation* (2).

To define rectangularity, let us first look at the case without ambiguity, i.e., if $\Pi = \{\pi\}$ and $\Phi^0_\Pi = \{\rho\}$ are singletons. After observing message m, the updated belief is given by ρ_m. Furthermore, the marginal beliefs of observing $m \in A \cup \tilde{A}$ under ρ is

$$\rho(\Omega, m) = \int_\Omega \rho(\omega, m)\,\mathrm{d}\omega = \int_\Omega p_0(\omega)\pi(m|\omega)\,\mathrm{d}\omega.$$

Then, the structure of Bayes' formula implies that multiplying the updated belief after observing message m with the marginal probability of observing m leads to the prior belief restricted to the events that the message is m. This holds for all messages m and, therefore, for all information sets of the partition defined above. Hence, integrating over all $m \in \operatorname{supp}(\Pi)$ leads to the prior belief ρ

$$\rho(\omega, m) = \int_{\operatorname{supp}(\Pi)} \rho(\Omega, m')\rho_{m'}(\omega, m)\,\mathrm{d}m',$$

where $\rho_{m'}(\omega, m) = 0$ if $m \neq m'$. Now, we generalize these considerations to an ambiguous setting, i.e., Π is not a singleton. Rectangularity requires that any combination of marginal beliefs and updated beliefs is a prior belief that the agent considers possible. The Receiver knows which messages he could receive and which updated beliefs potentially exist. Then, rectangularity requires that any combination of marginal and updated belief is an element of the ex-ante belief set.

Definition 2. *The pasting*

$$\bar{\rho} \circ (\rho_{\hat{m}})_{\hat{m}} : \Omega \times (A \cup \tilde{A}) \to [0,1]$$

of an ex-ante belief $\bar{\rho} \in \Phi^0_\Pi$ *and a collection of updated beliefs* $(\rho_{\hat{m}})_{\hat{m}} \in \bigtimes_{\hat{m} \in supp(\Pi)} Bay(\Phi^0_\Pi|\hat{m})$ *is defined as*[7]

$$\begin{aligned}\bar{\rho} \circ (\rho_{\hat{m}})_{\hat{m}}(\omega, m) &:= \int_{supp(\Pi)} \bar{\rho}(\Omega, \hat{m})\rho_{\hat{m}}(\omega, m)\,\mathrm{d}\hat{m} \\ &= \Big(\int_\Omega p_0(\omega')\bar{\pi}(m|\omega')\,\mathrm{d}\omega'\Big)\frac{p_0(\omega)\pi(m|\omega)}{\int_\Omega p_0(\omega')\pi(m|\omega')\,\mathrm{d}\omega'}.\end{aligned}$$

The set of ex-ante beliefs is called ***rectangular*** *(or stable under pasting) if it contains all pastings of an ex-ante belief* $\bar{\rho} \in \Phi^0_\Pi$ *and interim beliefs* $(\rho_{\hat{m}})_{\hat{m}}$, *i.e.,*

$$\bar{\rho} \circ (\rho_{\hat{m}})_{\hat{m}}(\cdot) \in \Phi^0_\Pi$$

[7]Please note, that the pasting is always well defined due to the common support assumption. Furthermore, the second equality follows since $\rho(\omega, m|\hat{m}) = 0$ if $m \neq \hat{m}$.

for all $\bar{\rho} \in \Phi^0_\Pi$ *and* $(\rho_{\hat{m}})_{\hat{m}} \in \bigtimes_{\hat{m} \in supp(\Pi)} Bay(\Phi^0_\Pi|\hat{m})$.

If Φ^0_Π is not rectangular, one can always construct the smallest set, which is rectangular and contains Φ^0_Π by backward induction. We call this set the rectangular hull and denote it with $\mathrm{rect}(\Phi^0_\Pi)$. Simple calculations show that $\mathrm{Bay}(\Phi^0_\Pi|\hat{m}) = \mathrm{Bay}(\mathrm{rect}(\Phi^0_\Pi)|\hat{m})$. The same holds for the set of marginal beliefs under Φ^0_Π and $\mathrm{rect}(\Phi^0_\Pi)$. For a more detailed explanation of the rectangular hull's construction and properties, please see Pahlke (2022) or Epstein and Schneider (2003).

So far, we focused on the beliefs of the Receiver. The Sender only chooses an action at the ex-ante stage. Therefore, the interim beliefs of the Sender do not influence the equilibria of the game. If the Sender does not know which communication device generated the message, her interim and ex-ante belief sets and the rectangular hull coincide with the Receiver's beliefs. However, even if $\Phi^0_\Pi \subsetneq \mathrm{rect}(\Phi^0_\Pi)$, the marginal beliefs of observing message m are the same for Φ^0_Π and $\mathrm{rect}(\Phi^0_\Pi)$. Thus, the ex-ante maximization problem of the Sender given $\mathrm{rect}(\Phi^0_\Pi)$ is the same as given Φ^0_Π.

Alternatively, we could define an information structure of the Sender that does not influence the ex-ante decision of the Sender but ensures that the ex-ante belief set of the Sender is rectangular for any Π. For example, the Sender could observe which communication device generated the observed message at the interim stage. If the Sender learns which communication device generated the message, Φ^0_Π is rectangular for all Π. By definition, rectangularity depends on the information structure faced by a player. Therefore, assuming heterogeneous information structures for Sender and Receiver would induce heterogeneous rectangular hulls. However, heterogeneous rectangular beliefs only arise due to heterogeneous information structures. Pahlke (2022) discusses the relation between information structures and common rectangular beliefs in more detail. However, the present paper aims to find a belief formation process that ensures dynamically consistent behavior. Since the Sender can never behave dynamically inconsistent, we do not go into details.

3 Dynamic Consistency and Perfect Bayesian Equilibrium

Finally, we show that rectangularity implies dynamically consistent behavior of the Receiver and, therefore, the existence of a Perfect Bayesian equilibrium.

Definition 3. *A Perfect Bayesian equilibrium with rectangular beliefs consists of a strategy* Π^* *of the Sender, a strategy* $(\hat{a}_m)_{m \in M}$ *of the Receiver, and a belief system* Ψ *for each player. Strategies and belief systems have to satisfy the following conditions:*

- *The belief systems of both players consist of an ex-ante belief set* Ψ^0_i *and interim belief set* Ψ^m_i *for each message* $m \in A \cup \tilde{A}$ *such that*

$$\Psi^0_R = \mathrm{rect}(\Phi^0_{\Pi^*})$$
$$\Psi^0_S = \Phi^0_{\Pi^*}.$$

Furthermore, the interim belief sets are derived by Bayes rule whenever possible, i.e., $\Psi_i^m = Bay(\Psi_i^0|m)$ for all $m \in \operatorname{supp}(\Pi^*)$.

- *The equilibrium strategy of the Sender Π^* with supp$(\Pi^*) \subseteq A \cup \tilde{A}$ maximizes his ex-ante worst-case expected utility*

$$\min_{\rho \in \Psi_S^0} \mathbb{E}_\rho \left[\nu(\hat{a}_m, \omega)\right].$$

- *The equilibrium strategy of the Receiver maximizes his interim worst-case expected utility for all $m \in \operatorname{supp}(\Pi^*)$*

$$\min_{\rho_m \in \Psi_R^m} \mathbb{E}_{\rho_m}(u(a_m, \omega))$$

and his ex-ante worst-case expected utility given the ex-ante belief set Ψ_R^0

$$\min_{\rho \in \Psi_R^0} \mathbb{E}_\rho(u(a_m, \omega)).$$

The following proposition shows that we can generalize any consistent planning equilibrium of BLL to a Perfect Bayesian equilibrium using rectangularity. Thus, the results of BLL can be generalized to a setting with dynamically consistent behavior. Therefore, our results show that the Sender can benefit from ambiguous communication even if the Receiver behaves dynamically consistently.

Proposition 2. *Let (M, Π) be the optimal ex-ante choice of the Sender and $(\hat{a}_m)_{m \in M}$ the optimal interim choice of the Receiver as in BLL. Then, there exists (M^*, Π^*), with $M^* \subseteq A \cup \tilde{A}$ and $|\Pi^*| = 2$ that generate the same value of the Sender as (M, Π). Furthermore, Π^*, $(\hat{a}_m)_{m \in M^*}$ and $\Psi_R^0 = \operatorname{rect}(\Phi_{\Pi^*}^0)$, $\Psi_S^0 = \Phi_{\Pi^*}^0$ and $(\Psi_i^m)_{m \in M^*} = (Bay(\Psi_i^0|m))_{m \in M^*}$ are a Perfect Bayesian equilibrium with rectangular beliefs.*

Proof. First, due to Proposition 1, there exists (M^*, Π^*), with $M^* \subseteq A \cup \tilde{A}$ and $|\Pi^*| = 2$ that generate the same value of the Sender as (M, Π). The proof of Proposition 1 shows that the Receiver chooses the same action given M or M^* in the sense that any two messages $m, m' \in M$ that are not synonyms of each other but induce the same optimal strategy, i.e., $\hat{a}_m = \hat{a}_{m'}$, are replaced by the same message $\bar{m} \in M^*$. Therefore, even if the message sets M and M^* are different, the Receiver's played actions do not change, and $(\hat{a}_m)_{m \in M^*}$ is induced by $(\hat{a}_m)_{m \in M}$.

Furthermore, the Sender never behaves dynamically inconsistently. We only have to show that the Receiver's interim best response of BLL is an interim and ex-ante best response given rectangular beliefs. Remember that $p_{\hat{m}}^{\pi_k}(\cdot) = \rho^k((\cdot, \hat{m})|\hat{m})$ for all $\hat{m} \in \operatorname{supp}(\Pi)$ and that the set of Bayesian updates given Φ_Π^0 or $\operatorname{rect}(\Phi_\Pi^0)$ are the same. Therefore, the interim best response given the state beliefs of BLL is an interim best response given rectangular beliefs, as well. Furthermore, we can rewrite the ex-ante expected utility of the Receiver as

$$\min_{\rho \in \mathrm{rect}(\Phi^0_{\Pi^*})} \int_{\mathrm{supp}(\Pi)} \rho(\Omega, \hat{m}) \mathbb{E}_{\rho_{\hat{m}}}(u(a_{\hat{m}}, \omega)) \, \mathrm{d}\hat{m},$$

where $\rho_{\hat{m}}$ is the Bayesian update of ρ given message $\hat{m}$. We first show the following relation of ex-ante and interim worst-case expected utility. Let ρ^* denote the ex-ante worst-case belief given rectangular beliefs. Then,

$$\begin{aligned} &\int_{\mathrm{supp}(\Pi^*)} \rho^*(\Omega, \hat{m}) \mathbb{E}_{\rho^*_{\hat{m}}}(u(a_{\hat{m}}, \omega)) \, \mathrm{d}\hat{m} \\ &\qquad = \int_{\mathrm{supp}(\Pi^*)} \rho^*(\Omega, \hat{m}) \min_{\rho_{\hat{m}} \in \mathrm{Bay}(\mathrm{rect}(\Phi^0_{\Pi^*})|\hat{m})} \mathbb{E}_{\rho_{\hat{m}}}(u(a_{\hat{m}}, \omega)) \, \mathrm{d}\hat{m}. \end{aligned} \tag{3}$$

To prove Equation 3, we first show that the left hand side is greater equal than the right hand side.

$$\begin{aligned} &\int_{\mathrm{supp}(\Pi^*)} \rho^*(\Omega, \hat{m}) \underbrace{\mathbb{E}_{\rho^*_{\hat{m}}}(u(a_{\hat{m}}, \omega))}_{\geq \min_{\rho_{\hat{m}} \in \mathrm{Bay}(\mathrm{rect}(\Phi^0_{\Pi^*})|\hat{m})} \mathbb{E}_{\rho_{\hat{m}}}(u(a_{\hat{m}}, \omega))} \mathrm{d}\hat{m} \\ \geq &\int_{\mathrm{supp}(\Pi^*)} \rho^*(\Omega, \hat{m}) \min_{\rho_{\hat{m}} \in \mathrm{Bay}(\mathrm{rect}(\Phi^0_{\Pi^*})|\hat{m})} \mathbb{E}_{\rho_{\hat{m}}}(u(a_{\hat{m}}, \omega)) \, \mathrm{d}\hat{m}. \end{aligned}$$

To prove the other direction, let $\rho'_{\hat{m}}$ be the worst-case belief given that he observed $\hat{m}$. Then, due to rectangularity, there exist $\bar{\rho} \in \mathrm{rect}(\Phi^0_{\Pi^*})$ such that $\rho^* \circ (\rho'_{\hat{m}})_{\hat{m}} = \bar{\rho}$. Furthermore rectangularity implies, that $\bar{\rho}(\cdot|\hat{m}) = \rho'(\cdot|\hat{m})$ and $\bar{\rho}(\Omega, \hat{m}) = \rho^*(\Omega, \hat{m})$ for all $\hat{m}$. Then,

$$\begin{aligned} \int_{\mathrm{supp}(\Pi^*)} \rho^*(\Omega, \hat{m}) \mathbb{E}_{\rho^*_{\hat{m}}}(u(a_{\hat{m}}, \omega)) \, \mathrm{d}\hat{m} &\leq \int_{\mathrm{supp}(\Pi^*)} \bar{\rho}(\Omega, \hat{m}) \mathbb{E}_{\bar{\rho}_{\hat{m}}}(u(a_{\hat{m}}, \omega)) \, \mathrm{d}\hat{m} \\ &= \int_{\mathrm{supp}(\Pi^*)} \rho^*(\Omega, \hat{m}) \mathbb{E}_{\rho'_{\hat{m}}}(u(a_{\hat{m}}, \omega)) \, \mathrm{d}\hat{m} \\ &= \int_{\mathrm{supp}(\Pi^*)} \rho^*(\Omega, \hat{m}) \min_{\rho_{\hat{m}} \in \mathrm{Bay}(\mathrm{rect}(\Phi^0_{\Pi^*})|\hat{m})} \mathbb{E}_{\rho_{\hat{m}}}(u(a_{\hat{m}}, \omega)) \, \mathrm{d}\hat{m}. \end{aligned}$$

Combining both directions proves Equation 3. Finally, we show that an interim best response of the Receiver is also an ex-ante best response. We denote with $\hat{a}_{\hat{m}}$ the (Sender preferred) interim best response of the Receiver given message $\hat{m}$, i.e.,

$$\min_{\rho_{\hat{m}} \in \mathrm{Bay}(\Phi^0_{\Pi^*}|\hat{m})} \mathbb{E}_{\rho_{\hat{m}}}(u(\hat{a}_{\hat{m}}, \omega)) \geq \min_{\rho_{\hat{m}} \in \mathrm{Bay}(\Phi^0_{\Pi^*}|\hat{m})} \mathbb{E}_{\rho_{\hat{m}}}(u(a_{\hat{m}}, \omega))$$

for any arbitrary $a_{\hat{m}} \in A$ and all $\hat{m} \in \mathrm{supp}(\Pi^*)$. We have to show that $(\hat{a}_{\hat{m}})_{\hat{m} \in \mathrm{supp}(\Pi^*)}$ is ex-ante optimal. Since $\rho(\Omega, \hat{m}) \geq 0$ for all $\hat{m} \in \mathrm{supp}(\Pi^*)$ and $\rho(\Omega, \hat{m}) = 0$ for all

$\hat{m} \notin \mathrm{supp}(\Pi^*)$, Equation 3 implies

$$\begin{aligned}
&\min_{\rho \in \mathrm{rect}(\Phi^0_{\Pi^*})} \int_{\mathrm{supp}(\Pi^*)} \rho(\Omega, \hat{m}) \mathbb{E}_{\rho_{\hat{m}}}(u(a_{\hat{m}}, \omega))\, \mathrm{d}\hat{m} \\
&\quad = \min_{\rho \in \mathrm{rect}(\Phi^0_{\Pi^*})} \int_{\mathrm{supp}(\Pi^*)} \rho(\Omega, \hat{m}) \min_{\rho'_{\hat{m}} \in \mathrm{Bay}(\Phi^0_{\Pi^*}|\hat{m})} \mathbb{E}_{\rho'_{\hat{m}}}(u(a_{\hat{m}}, \omega))\, \mathrm{d}\hat{m} \\
&\quad \leq \min_{\rho \in \mathrm{rect}(\Phi^0_{\Pi^*})} \int_{\mathrm{supp}(\Pi^*)} \rho(\Omega, \hat{m}) \min_{\rho'_{\hat{m}} \in \mathrm{Bay}(\Phi^0_{\Pi^*}|\hat{m})} \mathbb{E}_{\rho'_{\hat{m}}}(u(\hat{a}_{\hat{m}}, \omega))\, \mathrm{d}\hat{m} \\
&\quad = \min_{\rho \in \mathrm{rect}(\Phi^0_{\Pi^*})} \int_{\mathrm{supp}(\Pi^*)} \rho(\Omega, \hat{m}) \mathbb{E}_{\rho_{\hat{m}}}(u(\hat{a}_{\hat{m}}, \omega))\, \mathrm{d}\hat{m}
\end{aligned}$$

for any arbitrary $(a_{\hat{m}})_{\hat{m} \in \mathrm{supp}(\Pi)}$. Here the inequality follows from the interim optimality of $(\hat{a}_{\hat{m}})_{\hat{m} \in \mathrm{supp}(\Pi^*)}$ and the last equality from Equation 3.

Hence, the Receiver's ex-ante best response equals the interim best response, and the interim equilibrium of Beauchêne, Li, and Li (2019) satisfies ex-ante optimality. ■

BLL define a different version of rectangularity. In contrast to our setting, they define rectangularity for state beliefs, i.e., beliefs which are defined on the payoff-relevant state space Ω and not on the general state space $\Omega \times (A \cup \tilde{A})$. In their setting, rectangularity reduces ambiguity to risk. Therefore, they conclude that ambiguous communication cannot be profitable without dynamically inconsistent behavior. Our previous proposition shows this result does not hold if beliefs and rectangularity are defined on the general state space. Furthermore, with our definition of rectangularity we can extend all results from BLL to dynamically consistent behavior.

Remark 2. *The value of ambiguous persuasion as defined by the main result of BLL can be generalized to our setting with rectangular beliefs. Thus, whenever the Sender benefits from ambiguous communication in the setting of BLL, then there exists a Perfect Bayesian Equilibrium with rectangular beliefs such that the Sender has the same expected utility as in the setting of BLL. Hence, the gain of ambiguous communication, as characterized by BLL, does not rely on dynamically inconsistent behavior.*

To illustrate the previous results, we come back to our example from Section 2.2.

Example 2 (Example 1 cont.)**.** *Remember that the optimal ambiguous communication device was given by $\Pi = \{\pi, \pi'\}$ as depicted in Figure 4. Then, the Receiver's set of ex-ante beliefs is $\Phi^0_\Pi = \{\rho, \rho'\}$ with*

$$\rho(\omega, m) = \begin{cases} p_0 & \textit{if } m = n, \omega = \omega_n, \\ 1 - p_0 & \textit{if } m = p, \omega = \omega_p, \\ 0 & \textit{otherwise,} \end{cases} \qquad \rho'(\omega, m) = \begin{cases} p_0 & \textit{if } m = p, \omega = \omega_n, \\ 1 - p_0 & \textit{if } m = n, \omega = \omega_p, \\ 0 & \textit{otherwise,} \end{cases}$$

π	ω_n	ω_p
p	0	1
n	1	0

π'	ω_n	ω_p
p	1	0
n	0	1

Figure 4: Optimal Ambiguous Persuasion

and the rectangular hull is $\text{rect}(\Phi^0_\Pi) = \{\rho, \rho', \hat{\rho}, \bar{\rho}\}$ *where* ρ *and* ρ' *are as before and*

$$\bar{\rho}(\omega, m) = \begin{cases} p_0 & \text{if } m = p, \omega = \omega_p, \\ 1 - p_0 & \text{if } m = n, \omega = \omega_n, \\ 0 & \text{otherwise,} \end{cases} \qquad \hat{\rho}(\omega, m) = \begin{cases} p_0 & \text{if } m = n, \omega = \omega_p, \\ 1 - p_0 & \text{if } m = p, \omega = \omega_n, \\ 0 & \text{otherwise.} \end{cases}$$

Given the rectangular hull, the firm's ex-ante worst-case expected utility of always suspending the product is

$$\min_{\rho \in \text{rect}(\Phi^0_\Pi)} \rho(\omega_n, n) + \rho(\omega_n, p) = p_0 = 0.2$$

and his ex-ante worst-case expected utility of always launching the product is

$$\min_{\rho \in \text{rect}(\Phi^0_\Pi)} \rho(\omega_n, n)(-1) + \rho(\omega_n, p)(-1) + \rho(\omega_p, p) + \rho(\omega_p, n) = (-1)(1 - p_0) + p_0 = -0.6.$$

Thus, suspending the product is now ex-ante and interim optimal, and the Receiver behaves dynamically consistently.

Intuitively, rectangularity allows the Receiver to consider how ambiguous information might change his worst-case belief. The rectangular hull contains any combination of interim and ex-ante worst-case beliefs and, therefore, might include ex-ante beliefs that are not part of the original belief set.

By the construction of the rectangular hull, the marginal beliefs of messages under Φ^0_Π and $\text{rect}(\Phi^0_\Pi)$ are the same. Thus, the marginal probability of observing message m does not change when considering the rectangular hull. However, the marginal beliefs of the payoff-relevant state might change. In the worst-case belief, the Receiver might believe that an observed message m was generated by a communication device π when in fact, it was generated by a communication device π'. The rectangular hull considers this kind of misinterpretation and, therefore, can contain beliefs with new marginal probabilities of the payoff-relevant state.

More precisely, at the interim stage, which communication device generates the worst-case belief depends on the observed message and the action that the Receiver evaluates. Consider the previous example. Suppose the Receiver observes n and evaluates the expected utility of launching the product. Then, the worst case is that n was generated by π.

Thus, whenever the Receiver observes n and evaluates the utility of launching the product, he believes that n was generated by π. Therefore, he may misinterpret the message: if n is generated by π', but the Receiver believes that it was generated by π, then he concludes that the payoff-relevant state is ω_n even if the true state is ω_p. This misinterpretation occurs, if the communication device that generates the worst-case belief is not the true (but unknown) communication device that generated the message.

Ex-ante, the rectangular hull contains beliefs that consider misinterpretation. In the example, the rectangular hull contains the belief $\bar{\rho}$ with $\bar{\rho}(\omega_n, n) = 1 - p_0$. This belief represents exactly the misinterpretation described above: the Receiver believes that the message n was generated by π and therefore believes that the state is ω_n, although the message was generated by π' and the true state is ω_p. This misinterpretation implies that the marginal probability of the payoff-relevant state ω_n is $\sum_{m \in \{n,p\}} \bar{\rho}(\omega_n, m) = 1 - p_0$ and differs from the original prior state belief $\mathbb{P}(\omega_n) = p_0$.

In general, the rectangular hull contains any possible combination of misinterpretation, i.e., any belief where the Receiver believes that a message was generated by $\pi \in \Pi$ although it was generated by $\pi' \in \Pi$ with $\pi' \neq \pi$. Therefore, depending on the information structure, rectangularity can induce ex-ante beliefs with different marginal beliefs about the payoff-relevant state. Thus, rectangularity allows the Receiver to consider at the ex-ante stage the misinterpretation induced by the worst-case analysis at the interim stage. On the other hand, suppose he observes the message p. Then, the worst-case related to suspending the product is π, and the worst-case related to launching the product π'. The worst-case interpretation changes depending on the observed message and the action the Receiver is evaluating.

4 Further Models with Ambiguous Communication

A similar approach to defining rectangularity can be used in various models with ambiguous communication, e.g., in cheap talk or mechanism design. The main task is to define an adequate general state space and generalize beliefs to the general state space. To illustrate the general applicability of our results, we discuss the settings of Bose and Renou (2014) and Kellner and Le Quement (2018) in more detail.

4.1 Ambiguous Mechanism Design

Bose and Renou (2014) analyze a mechanism design setting with ambiguous communication. In their setting, there is a finite set of players N, a finite set of payoff-relevant types Θ_i for each player $i \in N$, and a finite set of alternatives X. Types are privately known, and there exists no exogenous ambiguity about types of opponents, i.e., ex-ante the distribution of types θ_{-i} is given by a singleton $p_i \in \Delta(\Theta_{-i})$ for player i. Players have maxmin preferences and update beliefs prior-by-prior.

Bose and Renou (2014) study the class of social choice functions $f : \times_{i \in N} \Theta_i \to X$ that is implementable by an ambiguous mechanism. An ambiguous mechanism consists of two steps: The second step, called the allocation mechanism, is a usual static mechanism specifying a finite set of messages M_i for each player and an allocation rule $g : \times_{i\in N} M_i \to X$. The first step adds an ambiguous communication device before the allocation mechanism is played. An ambiguous communication device consists of a finite set of messages that player i can send to the communication device $\hat{\Omega}_i$, a finite set of messages that player i can receive from the communication device Ω_i, and a set of probability systems Λ. The set of probability systems Λ corresponds to the set of communication devices Π in our setting. Hence, each $\lambda \in \Lambda$ specifies the probability that a profile of messages ω is received by the players given that they send the profile $\hat{\omega}$ to the communication device, i.e., $\lambda : \hat{\Omega} \to \Delta(\Omega)$, where $\hat{\Omega} = \times_{i\in N}\hat{\Omega}_i$ and $\Omega = \times_{i\in N}\Omega_i$.

They define a consistent planning equilibrium, i.e., players may behave dynamically inconsistently. However, similar to ambiguous persuasion, all their results can be extended to dynamically consistent players if players have rectangular beliefs. Here, the general state space is given by $\Theta \times \hat{\Omega} \times \Omega$.[8] Given an ambiguous communication device, the set of ex-ante beliefs of a type θ_i is

$$\Phi^0_\Lambda = \left\{ \phi \in \Delta(\Theta \times \hat{\Omega} \times \Omega) : \exists \lambda \in \Lambda \text{ s.t. } \phi(\hat{\theta}, \hat{\omega}, \omega) = \lambda(\hat{\omega})[\omega] p_i[\hat{\theta}_{-i}] \mathbb{1}_{\hat{\theta}_i = \theta_i} \right\}.$$

Now, we can define rectangularity analogously to Definition 2 and extend all results from Bose and Renou (2014) to dynamically consistent players. Thus, if a social choice function is implementable by an ambiguous mechanism of Bose and Renou (2014) for dynamically inconsistent players, then the same social choice function can be implemented by the same ambiguous mechanism for dynamically consistent players with rectangular beliefs.

4.2 Ambiguous Cheap Talk

Kellner and Le Quement (2018) study a cheap talk setting with ambiguous communication. They prove that an ambiguous strategy of the Sender can lead to a pareto improvement compared to the standard non-ambiguous cheap talk. Their setting is based on the standard non-ambiguous cheap talk setting of Crawford and Sobel (1982). The game consists of two players, a Sender, and a Receiver. The Sender has private information about a risky payoff-relevant state $\omega \in \Omega = [0, 1]$ and an ambiguous payoff-irrelevant state $\theta \in \Theta$. An Ellsbergian communication strategy is a standard communication strategy $q_\theta(\cdot|\omega) \in \Delta(M)$ for each $\theta \in \Theta$, where M is a finite message space. A strategy of the Receiver is a mapping $M \to \Delta(\mathbb{R})$. The Receiver's interim belief set is derived by updating the prior state belief p on Ω with respect to each communication strategy $q_\theta(\cdot|\omega)$.

As in the ambiguous persuasion setting, the equilibrium strategy of the Receiver is not ex-ante optimal. However, similarly to the procedure described above, defining beliefs and rectangularity over the general state space $\Omega \times \Theta$ leads to a Perfect Bayesian equilibrium

[8] Note, that $\hat{\Omega}$ and Ω are specified by the mechanism and not part of the strategy of the players. Therefore, an analog to Proposition 1 is not needed here.

with rectangular beliefs with the same strategies as in the interim equilibrium of Kellner and Le Quement (2018). Thus, ambiguous cheap talk can lead to a pareto improvement compared to the standard non-ambiguous cheap talk, even if players behave dynamically consistently.

5 Discussion

We study dynamically consistent behavior in an ambiguous persuasion setting. First, we show that restricting the message set to straightforward messages and synonyms is without loss of generality. Given this result, we can define beliefs over the more general state space $\Omega \times A \cup \tilde{A}$. This state space allows for dependence of the risky state and ambiguous signals. Therefore, the Receiver can consider the ambiguous information structure at the ex-ante stage. Then, rectangular beliefs ensure dynamically consistent behavior in ambiguous persuasion and the existence of a Perfect Bayesian equilibrium. Thus, ambiguity induces new equilibria in persuasion settings, even if the players behave dynamically consistently. To conclude, we discuss some related issues and literature.

Ex-ante preferences and commitment device Rectangularity allows players to take their future worst-case beliefs at the ex-ante stage into account and implies that the optimal interim actions become ex-ante optimal. Alternatively, one could assume that the Receiver could commit to his ex-ante optimal action to study dynamically consistent behavior. However, this requires a strong commitment device or a specific updating rule that allows players to ignore all interim beliefs contradicting the ex-ante worst-case belief. Hanany and Klibanoff (2007) propose such updating rules for maxmin preferences.

Concurrent with our work, Cheng (2021) shows that the Sender cannot gain from ambiguous persuasion if the Receiver can commit to his ex-ante optimal choice. The same results can be archived without commitment if the Receiver uses the updating rule of Hanany and Klibanoff (2007). Even if rectangularity and the updating rules of Hanany and Klibanoff (2007) lead to dynamically consistent behavior, they may induce different equilibria. The updating rules of Hanany and Klibanoff (2007) restrict the interim belief set to beliefs that maintain the ex-ante optimal choice interim optimal. In contrast, rectangularity enlarges the ex-ante belief set such that the interim optimal choice becomes ex-ante optimal.

Our example can illustrate the difference between the approaches. Using the updating rules of Hanany and Klibanoff (2007), the ex-ante belief set of the Receiver is Φ^0_Π. Then, ex-ante, he would prefer to launch the product for any message he could observe. After updating Φ^0_Π with the updating rules of Hanany and Klibanoff (2007), launching the product is still interim optimal for any message. Hence, given the updating rules of Hanany and Klibanoff (2007), the dynamically consistent Receiver would always launch the product, and the Sender cannot benefit from ambiguous persuasion. Given rectangularity, the Receiver's ex-ante belief set is given by the rectangular hull $\text{rect}(\Phi^0_\Pi)$ and always suspending the product is ex-ante and interim optimal. Hence, a dynamically consistent Receiver with rectangular beliefs will always suspend the product, and the Sender can gain from ambiguous persuasion.

Even if both approaches imply dynamically consistent behavior, the interpretation is different. A Receiver using the updating rules of Hanany and Klibanoff (2007) commits to his ex-optimal choice and ignores any information that would change his worst-case belief. On the other hand, a Receiver with rectangular beliefs considers that he will receive ambiguous information before deciding.

Value of Information Our work is related to the literature on the negative value of ambiguous information. The Receiver could prefer to ignore the ambiguous information and commit to his ex-ante optimal choice. Thus, the Receiver can have a negative value of information. However, the Receiver can also benefit from ambiguous information.

Ambiguous information induces two effects. On the one hand, an ambiguous communication device generates ambiguous beliefs and, therefore, decreases the worst-case expected utility of the Receiver. On the other hand, the communication device still reveals information about the state. The Receiver's value of information is negative if the first effect dominates the second effect.

Given the positive or negative value of information, a natural extension of the model would be to allow the Receiver to not listen to the Sender if the provided information is not valuable. This would force the Sender to choose a set of communication devices that provide a positive value of information for the Receiver. In Section C in the Appendix, we discuss the (negative) value of information for ambiguous persuasion in greater detail as well as a condition for and an example of a positive value of information.

Li (2020) studies the relation between ambiguity aversion and an aversion of (partial) information. She shows that an ambiguity averse decision maker(DM) with maxmin preferences is always (weakly) avers to partial information. However, Li (2020) assumes that the DM's set of acts is the same with and without ambiguous information. In contrast, our setting implies that given p_0, the DM can only choose from constant acts. Given an ambiguous communication device, the DM can choose any act which is measurable with respect to the information partition induced by the communication device. These are precisely the two effects we describe above. On the one hand, an ambiguous information device induces ambiguity, which decreases the utility of an ambiguity avers Receiver. On the other hand, anticipating this information at the ex-ante stage allows the Receiver to choose an action for each message that could occur with positive probability. Li (2020) focuses only on the first effect. Therefore, her result about partial information aversion of maxmin preferences does not imply that the Receiver's value of information is always negative in our setting.[9]

[9] The same consideration applies to the cheap talk setting of Kellner and Le Quement (2018).

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A Proofs

Proof of Proposition 1. Corollary 1 of BLL shows that there exists π_1 and π_2 such that $(M, \{\pi_1, \pi_2\})$ generates the same value as (M, Π). Hence, we have to show that (M', Π') generates the same value as $(M, \{\pi_1, \pi_2\})$. We first look at the case where the Sender does not use synonyms.

i) The Sender does not use synonyms.

   Since $(M, \{\pi_1, \pi_2\})$ does not use synonyms, there does not exist $m, m' \in M$ with $m \neq m'$ such that $\hat{a}_m = \hat{a}_{m'}$. Remember that p_m^π denotes the posterior state belief of the Receiver given the message m and the communication device π. Furthermore, $\hat{a}_m$ denotes Receivers best response given message $m \in M$ and the communication devices $\{\pi_1, \pi_2\}$. Since $(M, \{\pi_1, \pi_2\})$ does not use synonyms, there exists at most one $m \in M$ for each $a \in A$ such that $a = \hat{a}_m$. We define $\bar{\pi}_i(\cdot|\omega) \in \Delta M'$ with $M' = A$ such that

$$\bar{\pi}_i(a|\omega) = \begin{cases} \pi_i(m|\omega) & \text{if } \exists m \in M \text{ with } a = \hat{a}_m, \\ 0 & \text{otherwise.} \end{cases}$$

   Then, the posterior state belief $p_m^{\pi_i}$ equals the posterior state belief $p_a^{\bar{\pi}_i}$ if $a = \hat{a}_m$. Therefore, $(M, \{\pi_1, \pi_2\})$ and $(M', \{\bar{\pi}_1, \bar{\pi}_2\})$ generate the same set of posterior state beliefs and the same best response of the Receiver. Since the best response does not change, the value of the Sender is the same for both signals.

ii) Sender uses synonyms.

   If $(M, \{\pi_1, \pi_2\})$ uses synonyms, we can split M in M_1 and M_2 such that there exist a bijection between M_1 and M_2 and $M_1 \cup M_2 = M$. Then $(M_1, \{\hat{\pi}_1, \hat{\pi}_2\})$ with

$$\hat{\pi}_i(m|\omega) = \frac{\pi_i(m|\omega)}{\sum_{m \in M_1} \pi_i(m|\omega)}$$

   defines a signal that does not use synonyms. Thus, as in Case i), there exists $(M_1', \{\bar{\pi}_1, \bar{\pi}_2\})$ with $M_1' = A$ that generates the same value as $(M_1, \{\hat{\pi}_1, \hat{\pi}_2\})$. Similar one can define the restriction of π_i to M_2 and find $(M_2', \{\tilde{\pi}_1, \tilde{\pi}_1\})$ with $M_2' = \tilde{A}$, that generates the same value as M_2 and the restriction of π_i to M_2. Then, $(M', \{\pi_1', \pi_2'\})$ with $M' = M_1' \cup M_2'$ and

$$\pi_i'(a|\omega) = \begin{cases} \bar{\pi}_i(a|\omega) \sum_{m \in M_1} \pi_i(m|\omega) & \text{if } a \in A, \\ \tilde{\pi}_i(a|\omega) \sum_{m \in M_2} \pi_i(m|\omega) & \text{if } a \in \tilde{A}, \end{cases}$$

generates the same value as $(M, \{\pi_1, \pi_2\})$.

■

B Detailed Calculations and Extensions of the Example

B.1 Less-extreme Communication Devices

Consider the setting of Example 1 with the set of communication devices in Figure B.1. Suppose the firm observes message p. If p was generated by π, the updated state belief is $\mathbb{P}(\omega_n|p) = 0.1$. If p was generated by π', the updated state belief is $\mathbb{P}(\omega_n|p) = 0.6$. Thus, the set of interim state beliefs after observing message p is $\{0.1, 0.6\}$. Similarly, the set of interim state beliefs after observing message n is $\{0.1, 0.6\}$, as well. Given this set of interim beliefs, suspending the product is optimal after observing any message.

π	ω_n	ω_p
p	0.4	0.9
n	0.6	0.1

π'	ω_n	ω_p
p	0.6	0.1
n	0.4	0.9

Figure B.1: Less-extreme Communication Devices

More generally, suspending the product is optimal for any set of interim state beliefs $\{\underline{p}, \overline{p}\}$ with $\underline{p} \geq 1 - 2\overline{p}$. Thus, any set of communication devices that generates for each message a set of interim state beliefs satisfying the above condition implies that the firm always suspends the product.

B.2 Construction of the Rectangular Hull

To construct the rectangular hull of Example 2, we need to calculate all updated beliefs ρ_m after observing message $m \in \{n, p\}$

$$\rho_p(\omega, m) = \begin{cases} 1 & \text{if } m = p, \omega = \omega_p, \\ 0 & \text{otherwise,} \end{cases} \qquad \rho_n(\omega, m) = \begin{cases} 1 & \text{if } m = n, \omega = \omega_n, \\ 0 & \text{otherwise,} \end{cases}$$

$$\rho_p'(\omega, m) = \begin{cases} 1 & \text{if } m = p, \omega = \omega_n, \\ 0 & \text{otherwise,} \end{cases} \qquad \rho_n'(\omega, m) = \begin{cases} 1 & \text{if } m = n, \omega = \omega_p, \\ 0 & \text{otherwise,} \end{cases}$$

and marginal beliefs of observing message $m \in \{n, p\}$

$$\begin{aligned} \operatorname{marg}(\rho(\cdot, p)) &= 1 - p_0, & \operatorname{marg}(\rho(\cdot, n)) &= p_0, \\ \operatorname{marg}(\rho'(\cdot, p)) &= p_0, & \operatorname{marg}(\rho'(\cdot, n)) &= 1 - p_0. \end{aligned}$$

By combining any marginal and updated belief, we obtain the rectangular hull $\operatorname{rect}(\Phi_\Pi^0) = \{\rho, \rho', \hat{\rho}, \bar{\rho}\}$ where ρ and ρ' are as before and

$$\bar{\rho}(\omega, m) = \begin{cases} p_0 & \text{if } m = p, \omega = \omega_p, \\ 1 - p_0 & \text{if } m = n, \omega = \omega_n, \\ 0 & \text{otherwise,} \end{cases} \qquad \hat{\rho}(\omega, m) = \begin{cases} p_0 & \text{if } m = n, \omega = \omega_p, \\ 1 - p_0 & \text{if } m = p, \omega = \omega_n, \\ 0 & \text{otherwise.} \end{cases}$$

C Value of Information

With ambiguous persuasion, the Receiver can be better off by deciding based on p_0 and ignoring the ambiguous information. This result is consistent with the recent literature on the (negative) value of information under ambiguity, e.g., Li (2020) or Hill (2020). However, BLL show in their subsections 6.3 and 6.4 that the Receiver may benefit from listening to an ambiguous device. A similar result holds in our setting.

We denote with $U^0(a)$ the ex-ante expected utility of action a of the Receiver without any additional information, i.e.,

$$U^0(a) = \int_\Omega u(a, \omega) p_0(\omega) \, \mathrm{d}\omega.$$

Definition 4. *A communication device* Π *has a positive value of information for the Receiver if*

$$\max_{(a_m)_{m \in \operatorname{supp} \Pi} \in A^{|\operatorname{supp} \Pi|}} \min_{\rho \in \operatorname{rect}(\Phi_\Pi^0)} \mathbb{E}_\rho(u(a_m, \omega)) \geq \max_{a \in A} U^0(a).$$

Ambiguous information induces two effects. On the one hand, an ambiguous communication device generates ambiguous beliefs and, therefore, decreases the worst-case expected utility of the Receiver. On the other hand, the communication device still reveals information about the state. This information allows the Receiver to choose an action that better fits the state and increases his expected utility. Then, the value of information is positive if the second effect exceeds the negative effect of ambiguity and ambiguity aversion.

BLL say that a communication device satisfies a participation constrain if

$$\max_{(a_m)_{m \in \operatorname{supp} \Pi} \in A^{|\operatorname{supp} \Pi|}} \min_{\pi \in \Pi} \int_\Omega \int_M \pi(m|\omega) u(a_m, \omega) \, \mathrm{d}m \, p_0(\omega) \, \mathrm{d}\omega \geq \max_{a \in A} U^0(a).$$

They call this condition a participation constraint since it ensures that the Receiver is willing to pay attention to the communication device. If the participation constraint is not satisfied, the Receiver would be better off ignoring the communication device, ex-ante. Since $\Phi_{\Pi}^{0} \subseteq \mathrm{rect}(\Phi_{\Pi}^{0})$, it follows

$$\begin{aligned}
&\max_{(a_m)_{m\in\operatorname{supp}\Pi}\in A^{|\operatorname{supp}\Pi|}} \min_{\pi\in\Pi} \int_{\Omega}\int_{M} \pi(m|\omega)u(a_m,\omega)\,\mathrm{d}m\,p_0(\omega)\,\mathrm{d}\omega \\
&= \max_{(a_m)_{m\in\operatorname{supp}\Pi}\in A^{|\operatorname{supp}\Pi|}} \min_{\rho\in\Phi_{\Pi}^{0}} \mathbb{E}_{\rho}(u(a_m,\omega)) \\
&\geq \max_{(a_m)_{m\in\operatorname{supp}\Pi}\in A^{|\operatorname{supp}\Pi|}} \min_{\rho\in\mathrm{rect}(\Phi_{\Pi}^{0})} \mathbb{E}_{\rho}(u(a_m,\omega)).
\end{aligned}$$

Hence, any communication device with a positive value of information satisfies the participation constraint of BLL.

BLL characterize a condition that guarantees that the Receiver benefits from listening to a communication device (see BLL Proposition 8). We now translate this condition to our setting. We denote with a_0 the default actions, i.e., the action that maximizes $U^0(a)$.

Definition 5. *Let $\hat{a}_m$ denote the interim optimal action of the Receiver given the belief set Bay$(\mathrm{rect}(\Phi_{\Pi}^{0})|m)$. A message m is value-increasing (to the Receiver) if $\mathbb{E}_{\rho_m}(u(\hat{a},\omega)) \geq U^0(a_0)$ for all $\rho_m \in$ Bay$(\mathrm{rect}(\Phi_{\Pi}^{0})|m)$.*

BLL show that a communication device Π satisfies the participation constraint if Π uses only value-increasing messages. The next proposition proves a stronger and very intuitive result: A communication device that increases the worst-case expected utility of the Receiver for any message has a positive value of information.

Proposition 3. *If Π only uses value-increasing messages, then Π has a positive value of information for the Receiver.*

Proof. Since $\mathbb{E}_{\rho_m}(u(\hat{a},\omega)) \geq U^0(a_0)$ for all $\rho_m \in \mathrm{Bay}(\mathrm{rect}(\Phi_{\Pi}^{0})|m)$ it follows that

$$\min_{\rho_m\in\mathrm{Bay}(\mathrm{rect}(\Phi_{\Pi}^{0})|m)} \mathbb{E}_{\rho_m}(u(\hat{a},\omega)) \geq U^0(a_0). \tag{C.1}$$

Then, rectangularity and Equation C.1 imply

$$\begin{aligned}
&\max_{(a_m)_{m\in\operatorname{supp}\Pi}\in A^{|\operatorname{supp}\Pi|}} \min_{\rho\in\mathrm{rect}(\Phi_{\Pi}^{0})} \mathbb{E}_{\rho}(u(a_m,\omega)) \\
&= \min_{\rho\in\mathrm{rect}(\Phi_{\Pi^*}^{0})} \int_{\mathrm{supp}(\Pi^*)} \rho(\Omega,m) \min_{\rho'_m\in\mathrm{Bay}(\Phi_{\Pi^*}^{0}|m)} \mathbb{E}_{\rho'_m}(u(\hat{a}_m,\omega))\,\mathrm{d}m \\
&\geq \min_{\rho\in\mathrm{rect}(\Phi_{\Pi^*}^{0})} \int_{\mathrm{supp}(\Pi^*)} \rho(\Omega,m)U^0(a_0)\,\mathrm{d}m \\
&= U^0(a_0).
\end{aligned}$$

■

The following example illustrates an ambiguous communication device with a positive value of information for the Receiver.

Example 3. *Suppose the payoff-relevant state space consists of four states ω_1, ω_2, ω_3, and ω_4 and the common prior is $p_0(\omega) = \frac{1}{4}$ for all $\omega \in \Omega$. Further, the Receiver can choose between four actions: a, b, c, and d. The payoffs of the Sender and Receiver are given in Figure C.1. Without additional information, b would be the optimal action given the prior p_0.*

	ω_1	ω_2	ω_3	ω_4
a	$-1, 1$	$-4, -2$	$-9, -4$	$-16, -9$
b	$0, -1$	$-1, 0$	$-4, -1$	$-9, -4$
c	$-1, -4$	$0, -2$	$-1, 1$	$-4, -2$
d	$-4, -9$	$-1, -4$	$0, -1$	$-1, 0$

Figure C.1: Payoffs (S, R)

Consider the two communication devices in Figure C.2.

π	ω_1	ω_2	ω_3	ω_4
m_1	1	0	0	0
m_2	0	1	0	0
m_3	0	0	1	0
m_4	0	0	0	1

π'	ω_1	ω_2	ω_3	ω_4
m_1	0	1	0	0
m_2	1	0	0	0
m_3	0	0	0	1
m_4	0	0	1	0

Figure C.2: Ambiguous Communication Device

If the Receiver observes m_1 or m_2, he learns that the state is either ω_1 or ω_2 but perceives ambiguity about these two states. Similarly, if he observes m_3 or m_4, he learns that the state is either ω_3 or ω_4 but perceives ambiguity about these states. Then, with rectangular beliefs, the Receiver's interim and ex-ante optimal strategy is to choose b after observing m_1 or m_2 and d after observing m_3 or m_4. This implies a worst-case ex-ante expected utility of -1 for the Sender and the Receiver. Without any additional information, the Receiver would always choose b, and the ex-ante expected utility of the Sender and Receiver are -3.5 and -1.5, respectively. Hence, the ambiguous communication device is valuable for the Receiver and increases the expected utility of the Sender.